

Applying Decision-Making Principles to Synthesis of Weights in AHP

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Abstract

The Analytic Hierarchy Process (AHP) is a decision-making process which selects the best alternative having the highest score. Scores of alternatives are synthesized weights obtained from pairwise comparisons between alternatives' pairs under each criterion. In AHP, the synthesis of the weights is the sum of products of weights. It, however, seems unnatural that the summation of weights across the different criteria. We provide other syntheses based on four decision-making principles: Laplace's principle, Maximin principle, Hurwicz's principle, and Minimax regret principle.

1. Synthesis of weights in AHP

The growing of our societies makes choosing the best from alternatives be difficult. Many method for decision-making were developed in fields of applied mathematics and operations research. The Analytic Hierarchy Process (AHP) [1] is a process of such methods. The process, developed by Saaty in late of the 1970s, consists of three phases: constructing a hierarchical structure, rating weights of alternatives under each criterion, and synthesis the weights.

Hierarchical structures are typically three-layered. The top, the middle, and the bottom layer represent 'goal of the decision-making', 'criteria', and 'alternatives', respectively. Elements in the structure are evaluated from the top to the bottom.

In the rating phase, after deciding weights of criteria to achieve the goal, weights of alternatives are evaluated under each criterion. In AHP, the weights are derived from pairwise comparisons between alternatives.

Let the set of the alternatives in this article be $\{1, 2, \dots, m\}$. Under a criterion, for each pair of alternatives (i, j) , decision makers give a pairwise comparison value a_{ij} which represents how times

the alternative i is preferred than j . The pairwise comparison values, which are positive numbers, have reciprocal property; $a_{ij} = 1/a_{ji}$, and $a_{ii} = 1$. When the value a_{ij} is given, the alternative j is considered as it is $1/a_{ij}$ times preferred than i . The values are arranged into a pairwise comparison matrix A as follows:

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix},$$

where m is the number of alternatives. Weights of alternatives are derived from the matrix.

There are many methods to derive the weights from the matrix. The eigenvector method and the geometric mean method are often used among them. The eigenvector method adopts the principal eigenvector $w^* = [w_1^*, \dots, w_n^*]^T$ of the matrix as the weights; the i -th element w_i^* is the weight of the alternative i . The vector satisfies

$$Aw^* = \lambda_{\max} w^*,$$

where $\lambda_{\max}$ is the eigenvalue corresponding the principal eigen-

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vector. The geometric mean method adopts geometric means of each row of the matrix as the weights; the weight of the alternative i is $\overline{w_i} = (\prod_{j=1}^m a_{ij})^{\frac{1}{m}}$.

Since the weights represent just ratios between alternatives, the absolute value of each weight has nonsense. In the traditional AHP, the weights $\{w_1, \dots, w_m\}$ are normalized as $\sum_{i=1}^m w_i = 1$.

The weights of alternatives on all criteria are synthesized by the sum of products, and the synthesized value is referred to as *score* in this article. Let the set of criteria be $\{1, \dots, n\}$, the set of their weights for achieve the goal be $\{b_1, \dots, b_n\}$, and the set of weights of alternative under the criterion j be w_{ij} . Then the score of the alternative i is calculated as

$$s_i = \sum_{j=1}^n b_j w_{ij},$$

and the alternative with the highest score is chosen by the decision makers. Each term $b_j w_{ij}$, referred to as *local score* of the alternative i in this article, reflects rating of the alternative correctly under the criterion j .

The synthesis, however, is unnatural because added values are just only ratios. Some syntheses which seem to be more suitable than the sum of products can be considered. Of cause, scores are varied by which synthesis are used, then we need guidelines which synthesis have to use.

In this article, as basic research to build it, we provide how to synthesize weights under four principles: Laplace's principle, maximin principle, Hurwicz's principle, and minimax regret principle. Through calculations of a concrete example, we confirm, indeed, that scores on each principle, especially ranks of alternatives, are different.

2. Laplace's principle

On Laplace's principle, scores are interpreted as expected value with probabilities over local scores. When there is no information for which local score is preferred, it is valid that each local score has uniform probability $1/n$. That is based on the assumption referred to as Bernoulli's principle of insufficient reason or principle of Indifference [2].

The score of the alternative i on the principle is calculated as

$$s_i^L = \frac{1}{n} \sum_{j=1}^n b_j w_{ij}.$$

Notice that the normalized scores equal to scores by the synthesis of original AHP.

3. Maximin principle

On maximin principle, decision-makers must choose the alternative with the highest local score which is the lowest between local scores of the alternative. The score of the alternative i on the principle is calculated as

$$s_i^W = \min_{j=1, \dots, n} \{b_j w_{ij}\}.$$

4. Hurwicz's principle

The score of the alternative i on Hurwicz's principle [3] is a convex combination between the highest local score and the lowest local score, and is calculated as

$$s_i^H = \alpha \max_{j=1, \dots, n} \{b_j w_{ij}\} + (1 - \alpha) \min_{j=1, \dots, n} \{b_j w_{ij}\},$$

where $0 < \alpha < 1$. By interpreting how close the value α to 1 as optimism degree, the score is a balanced value between optimistic estimate and pessimistic estimate. The scores on the principle, of cause, depends on the value α . Decision makers must decide α before the synthesis according to their attitudes.

5. Minimax regret principle

When decision makers select an alternative whose score is not the highest, they will regret. Minimax regret principle [4] states that the decision makers must choose the alternative with the lowest the regret. The score of the alternative i on the principle is calculated as

$$s_i^S = - \max_{j=1, \dots, n} v_{ij},$$

where

$$v_{ij} = \max_{k=1, \dots, m} \{b_{kj} w_{kj}\} - b_j w_{ij}.$$

The value v_{ij} is the regret, often referred to as opportunity loss in microeconomics. It is the difference between choosing the alternative with the highest local score and choosing the alternative under each criterion j . Since the decision maker choose the alternative with the smallest regret and the regret must be greater than or equal to zero, the score is multiplied the regret by -1 .

6. An example

Let us consider the following example: the set of criteria is $\{1, 2\}$, the set of alternatives is $\{1, 2, 3\}$, and the pairwise comparison matrix of alternatives under the criterion 1 and the matrix under the criterion 2 are

$$A^1 = \begin{bmatrix} 1 & 3 & 5 \\ 1/3 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 1 & 1/3 \\ 1 & 1 & 1/3 \\ 3 & 3 & 1 \end{bmatrix}, \quad (1)$$

respectively. From the matrices, the normalized weights of alternatives under each criterion are calculated by the eigenvector method, whose results equal to by the geometric mean method in the example, and they are shown in Table 1. The value w_{ij} is the weight of the alternative i under the criterion j .

Table 1. Weights of alternatives from (1).

Alternative i	Weight w_{i1}	Weight w_{i2}
1	0.637	0.20
2	0.258	0.20
3	0.105	0.60

We consider how scores are changed by which the principle we adopt for three type of weights of criteria: $(b_1, b_2) = (0.8, 0.2)$, $(0.5, 0.5)$, and $(0.2, 0.8)$. The local scores corresponding the weights of criteria are shown in Table 2, 3, and 4. And for the weights, scores calculated on each principle are shown in Table 5, 6, and 7. The highest score on each principle is underlined. We can confirm that which the principle decision makers select varies the ranks of the alternatives. The ranks of scores on Hurwicz's principle, of cause, also varied by the value α . For example, for the case of the criteria's weight is $(0.8, 0.2)$ (Table 5), the alternative 1 is chosen when $\alpha > 0.1$, and the alternative 2 will be chosen otherwise.

Table 2. Local scores with criteria's weights $(0.8, 0.2)$.

Alternative i	0.8 w_{i1}	0.2 w_{i2}
1	0.510	0.040
2	0.206	0.040
3	0.084	0.120

Table 3. Local scores with criteria's weights $(0.5, 0.5)$.

Alternative i	0.5 w_{i1}	0.5 w_{i2}
1	0.319	0.100
2	0.129	0.100
3	0.053	0.300

Table 4. Local scores with criteria's weights $(0.2, 0.8)$.

Alternative i	0.2 w_{i1}	0.8 w_{i2}
1	0.127	0.160
2	0.052	0.160
3	0.021	0.480

Table 5. Scores with criteria's weights $(0.8, 0.2)$.

i	s_i^L	s_i^W	s_i^H	s_i^S
1	<u>0.547</u>	0.040	$0.470\alpha + 0.040$	<u>-0.080</u>
2	0.247	0.040	$0.202\alpha + 0.040$	-0.300
3	0.204	<u>0.084</u>	$0.036\alpha + 0.084$	-0.430

Table 6. Scores with criteria's weights $(0.5, 0.5)$.

i	s_i^L	s_i^W	s_i^H	s_i^S
1	<u>0.419</u>	<u>0.100</u>	$0.219\alpha + 0.100$	<u>-0.200</u>
2	0.229	<u>0.100</u>	$0.029\alpha + 0.100$	<u>-0.200</u>
3	0.303	0.053	$0.247\alpha + 0.053$	-0.270

Table 7. Scores with criteria's weights $(0.2, 0.8)$.

i	s_i^L	s_i^W	s_i^H	s_i^S
1	0.287	<u>0.127</u>	$0.033\alpha + 0.127$	-0.320
2	0.212	0.052	$0.108\alpha + 0.052$	-0.320
3	<u>0.501</u>	0.021	$0.459\alpha + 0.021$	<u>-0.110</u>

7. An approach from the dominant AHP

We remark again, these syntheses are operations between ratios under each criterion. It has nonsense when comparing ratios across another criterion, and causes some issues. Belton and Gear [5] pointed out that AHP has a rank reversal issue. Adding a copy alternative, whose weights are same as another alternative, may change the orders among some alternatives in the score order. It is not undesirable property when we consider the independence of irrelevant alternatives [6] or binary independence. The normalizing weights is the main cause of the rank reversal. Since the sum of weights is bound to 1, adding an alternative reduces relative weights of alternatives which is

same as the adding one. The syntheses on the decision-making principles also have the issue.

To avoid the rank reversal, Kinoshita and Nakanishi [7] advocated just comparing between a special alternative, called dominant alternative, and other alternatives instead of pairwise comparisons for all pairs of alternatives. In the improved AHP, called dominant AHP, the weight of alternative i under the criterion j from the dominant alternative d is rated as w_{ij}^d ; the alternative i is preferred w_{ij}^d times than d on the criteria j . Each criterion's weight is decided as importance of the criterion to achieve the goal by the dominant alternative. Let each weight of the criterion j on the alternative d be b_j^d , then the score of the alternative i is $\sum_{j=1}^n w_{ij}^d b_j^d$. The score of the dominant alternative d is 1 because $w_{dk}^d = 1, k = 1, ..., n$.

Let us consider local scores of the dominant AHP's approach with the dominant alternative 1, and suppose that relative weights under each criterion are not changed by rating in the dominant AHP; $w_{ij}^1 = w_{ij} / w_{1j}$. Then weights w_{ij}^1 and local scores $b_j^1 w_{ij}^1$ of the previous example are reformulated as in Table 8, 9, 10, and 11. Each weight b_j^1 is also supposed that $b_j^1 = b_j$ where b_j is criteria's weights in the example. And scores on each principle are shown in Table 12, 13, and 14. Even in the approach of the dominant AHP, different principles make different ranks of alternatives.

Table 8. Weights of alternatives.

Alternative i	Weight w_{i1}^1	Weight w_{i2}^1
1	1.000	1.000
2	0.405	1.000
3	0.165	3.000

Table 9. Local scores with criteria's weights (0.8, 0.2).

Alternative i	0.8 w_{i1}^1	0.2 w_{i2}^1
1	0.800	0.200
2	0.324	0.200
3	0.132	0.600

Table 10. Local scores with criteria's weights (0.5, 0.5).

Alternative i	0.5 w_{i1}^1	0.5 w_{i2}^1
1	0.500	0.500
2	0.203	0.500
3	0.082	1.500

Table 11. Local scores with criteria's weights (0.2, 0.8).

Alternative i	0.2 w_{i1}^1	0.8 w_{i2}^1
1	0.200	0.800
2	0.081	0.800
3	0.033	2.400

Table 12. Scores with criteria's weights (0.8, 0.2).

i	s_i^L	s_i^W	s_i^H	s_i^S
1	1.000	0.200	$0.600\alpha + 0.200$	-0.400
2	0.524	0.200	$0.124\alpha + 0.200$	-0.476
3	0.732	0.132	$0.468\alpha + 0.132$	-0.668

Table 13. Scores with criteria's weights (0.5, 0.5).

i	s_i^L	s_i^W	s_i^H	s_i^S
1	1.000	0.500	0.500	-1.000
2	0.703	0.203	$0.297\alpha + 0.203$	-1.000
3	1.582	0.0082	$1.418\alpha + 0.082$	-0.418

Table 14. Scores with criteria's weights (0.2, 0.8).

i	s_i^L	s_i^W	s_i^H	s_i^S
1	1.000	0.200	$0.600\alpha + 0.200$	-1.600
2	0.882	0.081	$0.719\alpha + 0.081$	-1.600
3	2.433	0.033	$2.367\alpha + 0.033$	-0.167

8. Conclusions

We introduced syntheses of weights for AHP on four decision-making principles. Since the synthesis of AHP is the sum between local scores different criteria, the synthesis does not satisfy the principle of Bentham's utilitarianism. So, other syntheses can be considered in AHP. Through the concrete example, for syntheses on four decision-making principles, we showed each synthesis derive different results, and which principle decision-makers select is important. For this, the properties of the syntheses have to be studied.

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