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DISCUSSION PAPER

Homogeneous N-miner Blockchain Gap Games

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Abstract

This paper examines the strategic delay of mining rig activation in Bitcoin's Proof of Work (PoW) system, where miners bear both capital and operational costs. As transaction fees grow in importance for miner revenues, delaying rig activation can enhance profitability (Carlsten et al., 2016). Building on the game-theoretic model by Tsabary and Eyal (2018) for their simulation approaches, we provide a rigorous, proof-based analysis, offering closed-form equilibrium solutions for n-player games with realistic cost and reward structures. The primary contribution is a novel method to derive these solutions for non-differentiable expected payoff functions, extending prior results (Noguchi, 2023).

Keywords: Blockchains; Cryptocurrency; Mining Gap; Game Theory; Nash Equilibrium

JEL classification: D62, D82

1 Introduction

Consensus protocols form the backbone of blockchain networks, ensuring decentralization, security, and trust. Each protocol adopts a unique approach to achieving these goals. Proof of Work (PoW), first introduced in Bitcoin (Nakamoto, 2008), remains the most widely recognized consensus mechanism. PoW requires miners to solve complex cryptographic puzzles using substantial computational power, which guarantees a high level of security and decentralization. However, its energy inefficiency has led to the development of alternative protocols. Proof of Stake (PoS), proposed by King and Nadal (2012), selects validators based on their cryptocurrency holdings, significantly reducing energy consumption. Yet, PoS introduces the potential for centralization, as wealthier participants can gain disproportionate control.

Building on this, Delegated Proof of Stake (DPoS) incorporates a governance layer, allowing token holders to elect validators, enhancing efficiency but potentially concentrating power among a small group of delegates (Larimer, 2014). Proof of Authority (PoA), commonly employed in private or permissioned blockchains, replaces anonymous miners with known, trusted validators, improving efficiency but compromising decentralization (De Angelis et al., 2018). Other protocols, such as Proof of Burn (PoB), which combines aspects of both PoW and PoS (Stewart, 2012), and Proof of Space (PoSpace), utilizing storage capacity rather than computational power (Dziembowski et al., 2015), offer additional variations aimed at reducing environmental impact. More recently, Proof of History (PoH), implemented in Solana, introduces a verifiable timeline of events to improve the speed and scalability of block validation, often in conjunction with PoS (Yakovenko, 2018). While PoW continues to stand out for its unmatched security and decentralization, newer protocols like PoS and PoH seek to address its limitations, particularly regarding energy consumption and scalability.

Despite these advancements, consensus mechanisms remain vulnerable to exploitation. Deviations in behavior can undermine the fairness and decentralization of both PoW and PoS systems. For example, selfish mining in PoW occurs when miners withhold blocks to create private chains, gaining an unfair advantage (Eyal and Sirer, 2018). PoS, on the other hand, is susceptible to stake grinding (Andreina et al., 2020) and strategic block timing (Schwarz-Schilling et al., 2023), which allow validators to manipulate block selection for personal gain. The nothing-at-stake problem (Buterin, 2014) further compromises PoS networks, allowing validators to support multiple chains without penalty. DPoS is vulnerable to delegate collusion (Qi et al., 2022), concentrating power among a few representatives, while PoA faces risks of cloning attacks (Ekparinya et al., 2019) in permissioned networks, exacerbating centralization. These vulnerabilities highlight the ongoing challenges of maintaining both decentralization and fairness in blockchain systems.

This paper focuses on a specific deviant behavior observed in Bitcoin's PoW system: the strategic delay in activating mining rigs. Miners face both capital costs for owning rigs and operational costs, such as electricity, for running them. By strategically delaying the activation of their rigs, miners can optimize costs, particularly in environments where mining profitability fluctuates. Blockchain networks like Bitcoin operate as decentralized, distributed ledgers, maintained by miners who validate transactions and compile them into blocks. Each block is cryptographically linked to its predecessor, forming a chain shared across all network participants. To add a new block, miners must solve a cryptographic puzzle, a process that requires significant computational resources, often referred to as hash power. Successful miners are rewarded with newly minted coins and transaction fees.

Bitcoin's system limits the total number of coins that can be created, and over time, block subsidies—the rewards in newly minted coins—decrease, making transaction fees an increasingly important component of miners' compensation. This transition from subsidy-based to fee-based rewards introduces new incentive dynamics (Carlsten et al., 2016), as miners may delay participation until sufficient fees accumulate, making mining profitable. This behavior, known as the formation of "mining gaps," reduces the total available hash power, weakening the network's security.

This paper examines the strategic delay of mining rig activation in Bitcoin's PoW system, where miners face both capital and operational costs. By delaying rig activation, miners can optimize profits, particularly as transaction fees become a more significant part of their revenues over time (Carlsten et al., 2016). Tsabary and Eyal (2018) modeled this behavior using game theory, showing how miners might delay activation for greater profitability. Building on their work, we offer a rigorous, proof-based analysis of this strategic behavior, presenting concrete, closed-form Nash-like equilibrium solutions for n -player games under practically meaningful cost and reward structures.

The primary technical contribution of this paper is a novel method to derive closed-form equilibrium solutions for potentially non-differentiable ex-ante expected payoff functions U_i, defined as integrals where the integrands–representing profit at time t–depend discontinuously on miners' rig start times s.

This discussion paper includes detailed computational results that extend our earlier findings (Noguchi, 2023).

The remainder of this paper is organized as follows. In Section 2, we define gap games. Section 3 covers the equilibrium analysis of homogeneous n-miner gap games. Section 4 states and proves the main theorem. Finally, Section 5 concludes the paper.

2 The gap game

In what follows, $\mathbb{R}$ denotes the set of real numbers, and for each subset $A \subset \mathbb{R}$, we denote the characteristic function of A by 1_A, where $1_A(t) \equiv \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{otherwise} \end{cases}$. For a real number $r \in \mathbb{R}$, we define $r^+ \equiv \max\{0, r\}$.

We consider a simplified version of blockchain gap games. Let $N = \{1, \cdots, n\}$ be the set of miners, where $i \in N$ represents an individual miner. Each miner i controls a single rig R_i and strategically sets a start time $s_i \in [0, \infty)$. At time t, R_i is said to be active if $s_i \leq t$. Let $B_i(s_i)$ denote the block finding time for rig R_i, omitting the dependency of s_i when clear from context.

As in Tsabary and Eyal (2018), we assume that the block finding time B_i for rig R_i is geometrically distributed; that is, the probability density function for the block-finding time B_i is

$$p_{B_i}(s_i, \mu; t) = 1_{[s_i, \infty)}(t)\, \mu e^{-\mu(t - s_i)},$$

and hence, its cumulative distribution function is

$$F_{B_i}(s_i, \mu; t) = 1 - e^{-\mu(t - s_i)^+},$$

where μ is the block creation rate to be determined by the protocol. Since the one-shot game ends when any miner discovers a new block, the overall block-finding time is a random variable $B = \min\{B_1, \cdots, B_n\}$.

Let λ_0 denote the base reward created and paid by the network protocol, and λ_1 the transaction fees accumulation rate. Thus, the expected income per miner at time t, given a start time profile $s = (s_1, \cdots, s_n)$, is

$$\rho_i(s; t)(\lambda_0 + \lambda_1 t),$$

where

$$\rho_i(s; t) = \begin{cases} 0 & \sum_{j=1}^{n} 1_{[s_j, \infty)}(t) = 0 \\ \frac{1_{[s_i, \infty)}(t)}{\sum_{j=1}^{n} 1_{[s_j, \infty)}(t)} & \sum_{j=1}^{n} 1_{[s_j, \infty)}(t) \neq 0 \end{cases}.$$

We remark that $\rho_i(s; t)$ represents the conditional probabilities of the successful mining of miner i, conditioned on the event that some miners successfully mine. In Tsabary, I., & Eyal, I. (2018, October), this ratio is simply taken as given without further probabilistic justification. Specifically, it is given as the ratio of the number of active rigs owned by miner i at time t to the total number of active rigs at time t. Thus, if each miner owns a single rig, as we assume for simplicity, then the ratio will be

$$\frac{1_{[s_i,\infty)}(t)}{\sum_{j=1}^{n} 1_{[s_j,\infty)}(t)},$$

provided that some miners activate their rigs no later than t. The detailed derivation of this ratio is provided in Appendix 1.

We introduce two types of costs per time unit for miners: The first, c_0^i, for owning the rig R_i, and the second, c_1^i, for having activated R_i. Then, the expected payoff per miner at time t, given a start time profile $s = (s_1, \cdots, s_n)$, is

$$u_i(s;t) = \rho_i(s;t)(\lambda_0 + \lambda_1 t) - \left[c_0^i t + c_1^i (t - s_i)^+\right].$$

Following the arguments in Tsabary and Eyal (2018), we calculate the probability density function $p_B(s, \mu; t)$ of B, assuming the random variables $B_1, \cdots, B_n$ are independent (see Ito, 1978). Let (Ω, P) be the underlying probability space, where the random variables $B_i \to \mathbb{R}$ are defined. For each B_i, let P^{B_i} denote the induced probability measure on $\mathbb{R}$. Then, for each Borel subset $V \subset \mathbb{R}$, we have

$$P^{B_i}(V) = P\left(B_i^{-1}(V)\right)$$

and in particular,

$$\begin{aligned} F_{B_i}(s_i, \mu; t) &= P\left(B_i^{-1}((-\infty, t])\right) \\ &= 1 - P\left(B_i^{-1}([t, \infty))\right). \end{aligned}$$

Thus, we obtain

$$P\left(B_i^{-1}([t,\infty))\right) = e^{-\mu(t-s_i)^+}.$$

and hence

$$\begin{aligned}
P\left(B^{-1}\left([t,\infty)\right)\right) &= P\left(\bigcap_{i=1}^{n} B_i^{-1}\left([t,\infty)\right)\right) \\
&\prod_{i=1}^{n} P\left(B_i^{-1}\left([t,\infty)\right)\right) \\
&= e^{-\mu\sum\limits_{i=1}^{n}(t-s_i)^+},
\end{aligned}$$

and hence the cumulative distribution function of $p_B(s,\mu;t)$ is

$$\begin{aligned}
F_B(s,\mu;t) &= 1 - P\left(B^{-1}\left([t,\infty)\right)\right) \\
&= 1 - e^{-\mu\sum\limits_{i=1}^{n}(t-s_i)^+}.
\end{aligned}$$

We claim

$$p_B(s,\mu;t) = \mu\sum_{i=1}^{n} 1_{[s_i,\infty)}(t)\, e^{-\mu\sum\limits_{j=1}^{n}(t-s_j)^+}.$$

The derivation of the above formula can be found in Tsabary and Eyal (2018) with different but equivalent notations. We must prove the following lemma.

Lemma 1 We have

$$\int_{-\infty}^{t} \mu\sum_{i=1}^{n} 1_{[s_i,\infty)}(t')\, e^{-\mu\sum\limits_{j=1}^{n}\left(t'-s_j\right)^+} dt' = 1 - e^{-\mu\sum\limits_{j=1}^{n}(t-s_j)^+}.$$

Proof. Since the above expression is invariant under the permutations of $(s_1,\cdots,s_n)$, we may assume $s_1 \le s_2 \le \cdots \le s_n$. We have

$$\sum_{i=1}^{n} 1_{[s_i,\infty)} = \sum_{i=1}^{n-1} i\cdot 1_{[s_i,s_{i+1})} + n\cdot 1_{[s_n,\infty)},$$

as its validity is shown in the Appendix. Note that the above equality holds even with equalities among s_i

since $1_{[s_i,\infty)} + 1_{[s_{i+1},\infty)} = 1_{[s_i,s_{i+1})} + 2 \cdot 1_{[s_{i+1},\infty)}$ is valid with $s_i = s_{i+1}$. Then

$$\begin{aligned}
&\int_{-\infty}^{t} \mu \sum_{i=1}^{n} 1_{[s_i,\infty)}(t')\, e^{-\mu \sum_{j=1}^{n}(t'-s_j)^+} dt' \\
&= \int_{-\infty}^{t} \mu \left[\sum_{i=1}^{n-1} i \cdot 1_{[s_i,s_{i+1})}(t') + n \cdot 1_{[s_n,\infty)}(t') \right] e^{-\mu \sum_{j=1}^{n}(t'-s_j)^+} dt' \\
&= \mu \sum_{i=1}^{n-1} i \int_{-\infty}^{t} 1_{[s_i,s_{i+1})}(t')\, e^{-\mu \sum_{j=1}^{i}(t'-s_j)} dt' + \mu n \int_{-\infty}^{t} 1_{[s_n,\infty)}(t')\, e^{-\mu \sum_{j=1}^{n}(t'-s_j)} dt'.
\end{aligned}$$

For $t \leq s_1$, we obtain

$$\begin{aligned}
\mu \sum_{i=1}^{n-1} i \int_{-\infty}^{t} 1_{[s_i,s_{i+1})}(t')\, e^{-\mu \sum_{j=1}^{i}(t'-s_j)} dt' + \mu n \int_{-\infty}^{t} 1_{[s_n,\infty)}(t')\, e^{-\mu \sum_{j=1}^{n}(t'-s_j)} dt' &= 0 \\
&= 1 - e^{-\mu \sum_{j=1}^{n}(t-s_j)^+}.
\end{aligned}$$

For $s_n \leq t$, we obtain

$$\begin{aligned}
&\mu \sum_{i=1}^{n-1} i \int_{-\infty}^{t} 1_{[s_i,s_{i+1})}(t')\, e^{-\mu \sum_{j=1}^{i}(t'-s_j)} dt' + \mu n \int_{-\infty}^{t} 1_{[s_n,\infty)}(t')\, e^{-\mu \sum_{j=1}^{n}(t'-s_j)} dt' \\
&= \mu \sum_{i=1}^{n-1} i \int_{s_i}^{s_{i+1}} e^{-\mu \sum_{j=1}^{i}(t'-s_j)} dt' + \mu n \int_{s_n}^{t} e^{-\mu \sum_{j=1}^{n}(t'-s_j)} dt' \\
&= \sum_{i=1}^{n-1} \left[e^{-\mu \sum_{j=1}^{i}(s_i-s_j)} - e^{-\mu \sum_{j=1}^{i}(s_{i+1}-s_j)} \right] + \left[e^{-\mu \sum_{j=1}^{n}(s_n-s_j)} - e^{-\mu \sum_{j=1}^{n}(t-s_j)} \right] \\
&= 1 - e^{-\mu \sum_{j=1}^{n-1}(s_n-s_j)} + e^{-\mu \sum_{j=1}^{n-1}(s_n-s_j)} - e^{-\mu \sum_{j=1}^{n}(t-s_j)} \\
&= 1 - e^{-\mu \sum_{j=1}^{n}(t-s_j)} \\
&= 1 - e^{-\mu \sum_{j=1}^{n}(t-s_j)^+}.
\end{aligned}$$

For $s_{i_0} \le t \le s_{i_0+1}$, we obtain

$$\begin{aligned}
&\mu \sum_{i=1}^{n-1} i \int_{-\infty}^{t} 1_{[s_i, s_{i+1})}(t') \, e^{-\mu \sum_{j=1}^{i} \left(t'-s_j\right)} dt' + \mu n \int_{-\infty}^{t} 1_{[s_n, \infty)}(t') \, e^{-\mu \sum_{j=1}^{n} \left(t'-s_j\right)} dt' \\
&= \mu \sum_{i=1}^{i_0-1} i \int_{s_i}^{s_{i+1}} e^{-\mu \sum_{j=1}^{i} \left(t'-s_j\right)} dt' + \mu i_0 \int_{s_{i_0}}^{t} e^{-\mu \sum_{j=1}^{i_0} \left(t'-s_j\right)} dt' \\
&= 1 - e^{-\mu \sum_{j=1}^{i_0-1} \left(s_{i_0}-s_j\right)} + e^{-\mu \sum_{j=1}^{i_0} \left(s_{i_0}-s_j\right)} - e^{-\mu \sum_{j=1}^{i_0} (t-s_j)} \\
&= 1 - e^{-\mu \sum_{j=1}^{i_0} (t-s_j)} \\
&= 1 - e^{-\mu \sum_{j=1}^{n} (t-s_j)^+}.
\end{aligned}$$

■

3 The equilibrium analysis of homogeneous n-miner gap games

For a given rate μ, we seek a Nash equilibrium s^* for the game $G = [N, (U_i), ([0, \infty))]$, where miner i acts as a player who chooses a strategy $s_i \in [0, \infty)$ while the other players choose strategies $s_{-i} \in [0, \infty)^{n-1}$, where s_{-i} is obtained by deleting s_i in s. We recall that the blockchain protocol requires the expectation of B to be equal to the expected block time interval, BTI, dictated by the protocol. We have

$$\begin{aligned}
E[\mu, s; B] &= \int_0^\infty t p_B(s, \mu; t) \, dt \\
&= \int_0^\infty e^{-\mu \sum_{j=1}^{n} (t-s_j)^+} dt.
\end{aligned}$$

We relegate the detailed proof to the Appendix 2.

A Nash equilibrium s^* given μ must satisfy the additional constraint

$$\mathrm{BTI} = E[\mu, s^*; B].$$

It is not immediately clear whether or not such a μ exists. We call a pair (s^*, μ^*) an **equilibrium** for the blockchain gap-game G if s^* is a Nash equilibrium given μ^* and if the pair satisfies the block interval condition. We call (s^*, μ^*) a **gap equilibrium** if some miners choose strictly positive starting times.

We define the ex-ante expected payoff function for miner i, given μ, as

$$U_i(s) = \int_0^\infty u_i(s;t)\, p_B(s;t)\, dt,$$

where μ is suppressed for simplicity.

Lemma 2 For each $s \in \mathbb{R}^n$ and $i = 1, \cdots, n$, the function $u_i(s;t)\, p_B(s;t)$ is integrable on $[0, \infty)$.

Proof. Let $s^m = \min_j s_j$. Since $\sum_{j=1}^n 1_{[s_j,\infty)}(t) = 0$ if and only if $t < s^m$, we obtain

$$\begin{aligned} U_i(s) &= \mu \int_{s^m}^\infty 1_{[s_i,\infty)}(t)\,(\lambda_0 + \lambda_1 t)\, e^{-\mu \sum_{j=1}^n (t-s_j)^+} dt \\ &\quad - \mu \int_{s^m}^\infty \left[c_0 t + c_1 (t-s_i)^+\right] \sum_{j=1}^n 1_{[s_j,\infty)}(t)\, e^{-\mu \sum_{j=1}^n (t-s_j)^+} dt. \end{aligned}$$

Since $s^m \leq s_i$, $e^{-\mu \sum_{j=1}^n (t-s_j)^+} \leq e^{-\mu (t-s_i)^+}$ and $\sum_{j=1}^n 1_{[s_j,\infty)}(t) \leq n$, we compute

$$\begin{aligned} \mu \int_{s^m}^\infty 1_{[s_i,\infty)}(t)\,(\lambda_0 + \lambda_1 t)\, e^{-\mu \sum_{j=1}^n (t-s_j)^+} dt &\leq \mu \int_{s_i}^\infty (\lambda_0 + \lambda_1 t)\, e^{-\mu(t-s_i)} dt < \infty \\ \mu \int_{s^m}^\infty c_0 t \sum_{j=1}^n 1_{[s_j,\infty)}(t)\, e^{-\mu \sum_{j=1}^n (t-s_j)^+} dt &\leq \mu c_0 n \int_{s^m}^\infty t e^{-\mu(t-s^m)} dt < \infty \\ \mu \int_{s^m}^\infty c_1 (t-s_i)^+ \sum_{j=1}^n 1_{[s_j,\infty)}(t)\, e^{-\mu \sum_{j=1}^n (t-s_j)^+} dt &\leq \mu c_1 n \int_{s^m}^\infty (t-s_i)^+ e^{-\mu(t-s_i)^+} dt \\ &= \mu c_1 n \int_{s_i}^\infty (t-s_i)\, e^{-\mu(t-s_i)} dt < \infty, \end{aligned}$$

from which our claim follows. ■

Let σ be a permutation on $\{1, 2, \cdots, n\}$ and define

$$s^\sigma = \left(s_{\sigma(1)}, \cdots, s_{\sigma(n)}\right).$$

Lemma 3 For each σ, we have $U_i(s^\sigma) = U_{\sigma(i)}(s)$.

Proof. We express the integrand of $U_i(s)$ as $f(s_i, s; t)$. Then, we have $U_i(s) = \int_{s^m}^\infty f(s_i, s; t)\, dt$, where $f\left(s_{\sigma(i)}, s^\sigma; t\right) = f\left(s_{\sigma(i)}, s; t\right)$ holds for all σ. We also have $(s^\sigma)^m = \min_j s_{\sigma(j)} = \min_j s_j = s^m$ for all σ. We deduce

$$
\begin{aligned}
U_i\left(s^\sigma\right) &= \int_{(s^\sigma)^m}^{\infty} f\left(s_{\sigma(i)}, s^\sigma; t\right) dt \\
&= \int_{s^m}^{\infty} f\left(s_{\sigma(i)}, s; t\right) dt \\
&= U_{\sigma(i)}\left(s\right),
\end{aligned}
$$

as desired. ■

Let $D_+ = \mathbb{R}^n_+$. For each σ, define $D^{s^\sigma}_+ = \left\{s \in \mathbb{R}^n_+ : s_{\sigma(1)} \leq \cdots \leq s_{\sigma(n)}\right\}$, where $D^s_+ = \left\{s \in \mathbb{R}^n_+ : s_1 \leq \cdots \leq s_n\right\}$. Let $U_i^{s^\sigma} : \mathbb{R}^n \to \mathbb{R}$ be a function such that $U_i^{s^\sigma} \mid_{D^{s^\sigma}_+} = U_i \mid_{D^{s^\sigma}_+}$.

Lemma 4 For each σ, we have $U_i^s\left(s^\sigma\right) = U^{s^\sigma}_{\sigma(i)}\left(s\right)$ for all $s \in D^{s^\sigma}_+$. (May omit this)

Proof. Note that $s^\sigma \in D^s_+$ if and only if $s \in D^{s^\sigma}_+$. Let $s \in D^{s^\sigma}_+$. Then, since $s^\sigma \in D^s_+$, we reduce

$$
\begin{aligned}
U_i^s\left(s^\sigma\right) &= U_i^s \mid_{D^s_+} \left(s^\sigma\right) \\
&= U_i \mid_{D^s_+} \left(s^\sigma\right) \\
&= U_i\left(s^\sigma\right) \\
&= U_{\sigma(i)}\left(s\right) \\
&= U_{\sigma(i)} \mid_{D^{s^\sigma}_+} \left(s\right) \\
&= U^{s^\sigma}_{\sigma(i)}\left(s\right).
\end{aligned}
$$

■

Lemma 5 For each σ such that $\sigma^2 = \mathrm{id}$, we have $U_i^{s^\sigma}\left(s^\sigma\right) = U^s_{\sigma(i)}\left(s\right)$ for all $s \in D^s_+$.

Proof. Note that $s \in D^s_+$ if and only if $s^\sigma \in D^{s^\sigma}_+$ since $\sigma^2 = \mathrm{id}$. Let $s \in D^{s^\sigma}_+$. Then, since $s^\sigma \in D^{s^\sigma}_+$, we reduce

$$\begin{aligned}
U_i^{s^\sigma}\left(s^\sigma\right) &= U_i^{s^\sigma}\mid_{D_+^{s^\sigma}}\left(s^\sigma\right) \\
&= U_i\mid_{D_+^{s^\sigma}}\left(s^\sigma\right) \\
&= U_i\left(s^\sigma\right) \\
&= U_{\sigma(i)}\left(s\right) \\
&= U_{\sigma(i)}\mid_{D_+^{s}}\left(s\right) \\
&= U_{\sigma(i)}^{s}\left(s\right).
\end{aligned}$$

■

Proposition 6 Let $(\xi,\cdots,\xi)\in D_+$ be a Nash equilibrium for $\left\{U_i^s\mid_{D_+}\right\}$. Then, it is a Nash equilibrium for $\left\{U_i^{s^\sigma}\mid_{D_+}\right\}$ all σ such that $\sigma^2=\mathrm{id}$.

Proof. Suppose the conclusion is false. Then, there must be σ, i , and $a\in[0,\infty)$ such that $U_i^{s^\sigma}(\xi,\cdots,\xi)<U_i^{s^\sigma}\left(\xi,\cdots,\overset{i}{a},\cdots,\xi\right)$.

Since $U_i^{s^\sigma}\mid_{D_+}(\xi,\cdots,\xi)=U_{\sigma(i)}^{s}\mid_{D_+}(\xi,\cdots,\xi)$ and

$$\begin{aligned}
&U_i^{s^\sigma}\mid_{D_+}\left(\xi,\cdots,\overset{i}{a},\cdots,\xi\right)=U_i^{s^\sigma}\mid_{D_+}\left(\left(\xi,\cdots,\overset{\sigma(i)}{a},\cdots,\xi\right)^\sigma\right) \\
&\quad=U_{\sigma(i)}^{s}\mid_{D_+}\left(\xi,\cdots,\overset{\sigma(i)}{a},\cdots,\xi\right),
\end{aligned}$$

we reach a contradiction. ■

Proposition 7 Let $(\xi,\cdots,\xi)\in D_+$ be a Nash equilibrium for $\left\{U_i^s\mid_{D_+}\right\}$. Then, it is a Nash equilibrium for $\{U_i\}$.

Proof. Suppose the conclusion is false. Then, there must be i and $a\in[0,\infty)$ such that $U_i(\xi,\cdots,\xi)<U_i\left(\xi,\cdots,\overset{i}{a},\cdots,\xi\right)$. Without loss of generality, we may assume $\xi<a$. Let $\sigma=(i,n)$, where $\sigma^2=\mathrm{id}$. Then, we deduce

$$U_i\left(\xi,\cdots,\overset{i}{a},\cdots,\xi\right)=U_i^{s^\sigma}\mid_{D_+}\left(\xi,\cdots,\overset{i}{a},\cdots,\xi\right)$$

and

$$U_i\left(\xi,\cdots,\xi\right)=U_i^{s^\sigma}\mid_{D_+}\left(\xi,\cdots,\xi\right),$$

which shows that $(\xi,\cdots,\xi)\in D_+$ is not a Nash equilibrium for $\left\{U_i^{s^\sigma}\mid_{D_+}\right\}$, contradicting the premise of the proposition. ■

In the sequel, we will find a Nash equilibrium, given $\mu>0$, of in the form $(\xi,\cdots,\xi)\in D_+$ for $\{U_i\}$ by finding one for $\left\{U_i^s\mid_{D_+}\right\}$. To this end, it suffices to find a Nash equilibrium, given $\mu>0$, of in the form $(\xi,\cdots,\xi)\in D_+$ for $\{U_i^s\}$, where each U_i^s is defined on the entire $\mathbb{R}^n$.

Let $s_1\leq s_2\leq\cdots\leq s_n$ and $s_{n+1}=\infty$ Then, we obtain

$$\rho_i\left(s;t\right)=\sum_{j=i}^{n}\frac{1}{j}1_{[s_j,s_{j+1})}\left(t\right)$$

and

$$p_B\left(s;t\right)=\mu\sum_{j=1}^{n}j1_{[s_j,s_{j+1})}\left(t\right)e^{-\mu[(t-s_1)+\cdots+(t-s_j)]}.$$

We compute

$$
\begin{aligned}
U_i^s(s) &= \int_0^\infty u_i(s;t)\, p_B(s;t)\, dt \\
&= \int_0^\infty \left\{ \sum_{k=i}^n \frac{1}{k} 1_{[s_k,s_{k+1})}(t)(\lambda_0+\lambda_1 t) - \left[c_0 t + c_1 (t-s_i)^+\right] \right\} \mu \sum_{j=1}^n j 1_{[s_j,s_{j+1})}(t)\, e^{-\mu[(t-s_1)+\cdots+(t-s_j)]} dt \\
&= \mu \sum_{j=1}^n j \int_{s_j}^{s_{j+1}} \sum_{k=i}^n \frac{1}{k} 1_{[s_k,s_{k+1})}(t)(\lambda_0+\lambda_1 t)\, e^{-\mu[(t-s_1)+\cdots+(t-s_j)]} dt \\
&- \mu \sum_{j=1}^n j \int_{s_j}^{s_{j+1}} \left[c_0 t + c_1 (t-s_i)^+\right] e^{-\mu[(t-s_1)+\cdots+(t-s_j)]} dt \\
&= \mu \sum_{k=i}^n \sum_{j=1}^n \frac{j}{k} \int_{s_j}^{s_{j+1}} 1_{[s_k,s_{k+1})}(t)(\lambda_0+\lambda_1 t)\, e^{-\mu[(t-s_1)+\cdots+(t-s_j)]} dt \\
&- \mu \sum_{j=1}^n j \int_{s_j}^{s_{j+1}} \left[c_0 t + c_1 (t-s_i)^+\right] e^{-\mu[(t-s_1)+\cdots+(t-s_j)]} dt \\
&= \mu \sum_{k=i}^n \int_{s_k}^{s_{k+1}} (\lambda_0+\lambda_1 t)\, e^{-\mu[(t-s_1)+\cdots+(t-s_k)]} dt - \mu \sum_{j=1}^n j \int_{s_j}^{s_{j+1}} c_0 t e^{-\mu[(t-s_1)+\cdots+(t-s_j)]} dt \\
&- \mu \sum_{j=1}^n j \int_{s_j}^{s_{j+1}} c_1 (t-s_i)^+ \, e^{-\mu[(t-s_1)+\cdots+(t-s_j)]} dt \\
&= \mu \sum_{j=i}^n \int_{s_j}^{s_{j+1}} (\lambda_0+\lambda_1 t)\, e^{-\mu[(t-s_1)+\cdots+(t-s_j)]} dt - \mu \sum_{j=1}^n j \int_{s_j}^{s_{j+1}} c_0 t e^{-\mu[(t-s_1)+\cdots+(t-s_j)]} dt \\
&- \mu \sum_{j=i}^n j \int_{s_j}^{s_{j+1}} c_1 (t-s_i)\, e^{-\mu[(t-s_1)+\cdots+(t-s_j)]} dt
\end{aligned}
$$

and hence,

$$
\begin{aligned}
U_i^s\left(s_i,\xi_{-i}\right) &= \int_0^\infty u_i\left(s_i,\xi_{-i};t\right)p_B\left(s_i,\xi_{-i};t\right)dt \\
&= \mu\int_{s_i}^{\xi}\left(\lambda_0+\lambda_1 t\right)e^{-\mu[(i-1)(t-\xi)+(t-s_i)]}dt+\mu\int_{\xi}^{\infty}\left(\lambda_0+\lambda_1 t\right)e^{-\mu[(n-1)(t-\xi)+(t-s_i)]}dt \\
&- \mu i\int_{s_i}^{\xi}c_0 te^{-\mu[(i-1)(t-\xi)+(t-s_i)]}dt-\mu\left(i-1\right)\int_{\xi}^{s_i}c_0 te^{-\mu[(i-1)(t-\xi)]}dt-\mu n\int_{\xi}^{\infty}c_0 te^{-\mu[(n-1)(t-\xi)+(t-s_i)]}dt \\
&- \mu i\int_{s_i}^{\xi}c_1\left(t-s_i\right)e^{-\mu[(i-1)(t-\xi)+(t-s_i)]}dt-\mu n\int_{\xi}^{\infty}c_1\left(t-s_i\right)e^{-\mu[(n-1)(t-\xi)+(t-s_i)]}dt \\
&= \mu\int_{s_i}^{\xi}\left(\lambda_0+\lambda_1 t\right)e^{-\mu[it-(i-1)\xi-s_i]}dt-\mu i\int_{s_i}^{\xi}c_0 te^{-\mu[it-(i-1)\xi-s_i]}dt-\mu i\int_{s_i}^{\xi}c_1\left(t-s_i\right)e^{-\mu[it-(i-1)\xi-s_i]}dt \\
&+ \mu\left(i-1\right)\int_{s_i}^{\xi}c_0 te^{-\mu[(i-1)t-(i-1)\xi]}dt \\
&+ \mu\int_{\xi}^{\infty}\left(\lambda_0+\lambda_1 t\right)e^{-\mu[(n-1)(t-\xi)+(t-s_i)]}dt-\mu n\int_{\xi}^{\infty}c_0 te^{-\mu[nt-(n-1)\xi-s_i]}dt \\
&- \mu n\int_{\xi}^{\infty}c_1\left(t-s_i\right)e^{-\mu[nt-(n-1)\xi-s_i]}dt \\
&= \mu e^{\mu[(i-1)\xi+s_i]}\int_{s_i}^{\xi}\left[\left(\lambda_0+\lambda_1 t\right)-ic_0 t-ic_1\left(t-s_i\right)\right]e^{-\mu it}dt+\mu\left(i-1\right)e^{\mu(i-1)\xi}\int_{s_i}^{\xi}c_0 te^{-\mu(i-1)t}dt \\
&+ \mu e^{\mu[(n-1)\xi+s_i]}\int_{\xi}^{\infty}\left[\left(\lambda_0+\lambda_1 t\right)-nc_0 t-nc_1\left(t-s_i\right)\right]e^{-\mu nt}dt \\
&= \mu e^{\mu[(i-1)\xi+s_i]}\int_{s_i}^{\xi}\left[\left(\lambda_0+\lambda_1 t\right)-ic_0 t-ic_1\left(t-s_i\right)\right]e^{-\mu it}dt+\mu\left(i-1\right)e^{\mu(i-1)\xi}\int_{s_i}^{\xi}c_0 te^{-\mu(i-1)t}dt \\
&+ \mu\left(\lambda_0+nc_1 s_i\right)e^{\mu[(n-1)\xi+s_i]}\int_{\xi}^{\infty}e^{-\mu nt}dt+\mu\left(\lambda_1-nc_0-nc_1\right)e^{\mu[(n-1)\xi+s_i]}\int_{\xi}^{\infty}te^{-\mu nt}dt \\
&= \mu\left(\lambda_1-ic_0-ic_1\right)e^{\mu[(i-1)\xi+s_i]}\int_{s_i}^{\xi}te^{-\mu it}dt+\mu e^{\mu[(i-1)\xi+s_i]}\int_{s_i}^{\xi}\left[\lambda_0+ic_1 s_i\right]e^{-\mu it}dt \\
&+ \mu\left(i-1\right)e^{\mu(i-1)\xi}\int_{s_i}^{\xi}c_0 te^{-\mu(i-1)t}dt+\mu\left(\lambda_0+nc_1 s_i\right)e^{\mu[(n-1)\xi+s_i]}\int_{\xi}^{\infty}e^{-\mu nt}dt \\
&+ \mu\left(\lambda_1-nc_0-nc_1\right)e^{\mu[(n-1)\xi+s_i]}\int_{\xi}^{\infty}te^{-\mu nt}dt
\end{aligned}
$$

$$
\begin{aligned}
&U_i^s\left(s_i, \xi_{-i}\right) \\
&= \mu\left(\lambda_1 - ic_0 - ic_1\right) e^{\mu[(i-1)\xi + s_i]}\left(\frac{s_i e^{-i\mu s_i} - \xi e^{-i\mu\xi}}{i\mu} + \frac{e^{-i\mu s_i} - e^{-i\mu\xi}}{i^2\mu^2}\right) \\
&+ \mu\left(\lambda_0 + ic_1 s_i\right) e^{\mu[(i-1)\xi + s_i]}\left(\frac{e^{-i\mu s_i} - e^{-i\mu\xi}}{i\mu}\right) \\
&+ \mu c_0\left(i-1\right) e^{\mu(i-1)\xi}\left(\frac{s_i e^{-(i-1)\mu s_i} - \xi e^{-(i-1)\mu\xi}}{\left(i-1\right)\mu} + \frac{e^{-(i-1)\mu s_i} - e^{-(i-1)\mu\xi}}{\left(i-1\right)^2\mu^2}\right) \\
&+ \mu\left(\lambda_0 + nc_1 s_i\right) e^{\mu[(n-1)\xi + s_i]}\frac{e^{-n\mu\xi}}{n\mu} + \mu\left(\lambda_1 - nc_0 - nc_1\right) e^{\mu[(n-1)\xi + s_i]}\left(\frac{\xi e^{-n\mu\xi}}{n\mu} + \frac{e^{-n\mu\xi}}{n^2\mu^2}\right) \\
&= \mu\left(\lambda_1 - ic_0 - ic_1\right)\frac{s_i e^{\mu[(i-1)\xi - (i-1)s_i]}}{i\mu} - \mu\left(\lambda_1 - ic_0 - ic_1\right)\frac{\xi e^{\mu(-\xi + s_i)}}{i\mu} + \mu\left(\lambda_1 - ic_0 - ic_1\right)\frac{e^{\mu[(i-1)\xi - (i-1)s_i]}}{i^2\mu^2} \\
&- \mu\left(\lambda_1 - ic_0 - ic_1\right)\frac{e^{\mu(-\xi + s_i)}}{i^2\mu^2} + \mu\left(\lambda_0 + ic_1 s_i\right)\frac{e^{\mu[(i-1)\xi - (i-1)s_i]}}{i\mu} - \mu\left(\lambda_0 + ic_1 s_i\right)\frac{e^{\mu(-\xi + s_i)}}{i\mu} \\
&+ \mu c_0\left(i-1\right)\frac{s_i e^{-(i-1)\mu(s_i - \xi)}}{\left(i-1\right)\mu} - \mu c_0\left(i-1\right)\frac{\xi}{\left(i-1\right)\mu} + \mu c_0\left(i-1\right)\frac{e^{-(i-1)\mu(s_i - \xi)}}{\left(i-1\right)^2\mu^2} - \mu c_0\left(i-1\right)\frac{1}{\left(i-1\right)^2\mu^2} \\
&+ \mu\left(\lambda_0 + nc_1 s_i\right)\frac{e^{\mu(-\xi + s_i)}}{n\mu} + \mu\left(\lambda_1 - nc_0 - nc_1\right)\frac{\xi e^{\mu(-\xi + s_i)}}{n\mu} + \mu\left(\lambda_1 - nc_0 - nc_1\right)\frac{e^{\mu(-\xi + s_i)}}{n^2\mu^2} \\
&= \left(\lambda_1 - ic_0 - ic_1\right)\frac{s_i e^{\mu[(i-1)\xi - (i-1)s_i]}}{i} - \left(\lambda_1 - ic_0 - ic_1\right)\frac{\xi e^{\mu(-\xi + s_i)}}{i} + \left(\lambda_1 - ic_0 - ic_1\right)\frac{e^{\mu[(i-1)\xi - (i-1)s_i]}}{i^2\mu} \\
&- \left(\lambda_1 - ic_0 - ic_1\right)\frac{e^{\mu(-\xi + s_i)}}{i^2\mu} + \left(\lambda_0 + ic_1 s_i\right)\frac{e^{\mu[(i-1)\xi - (i-1)s_i]}}{i} - \left(\lambda_0 + ic_1 s_i\right)\frac{e^{\mu(-\xi + s_i)}}{i} \\
&+ c_0 s_i e^{-(i-1)\mu(s_i - \xi)} - c_0\xi + c_0\frac{e^{-(i-1)\mu(s_i - \xi)}}{\left(i-1\right)\mu} - c_0\frac{1}{\left(i-1\right)\mu} + \left(\lambda_0 + nc_1 s_i\right)\frac{e^{\mu(-\xi + s_i)}}{n} \\
&+ \left(\lambda_1 - nc_0 - nc_1\right)\frac{\xi e^{\mu(-\xi + s_i)}}{n} + \left(\lambda_1 - nc_0 - nc_1\right)\frac{e^{\mu(-\xi + s_i)}}{n^2\mu}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial s_i}U_i^s\left(s_i, \xi_{-i}\right) \\
&= \left(\lambda_1 - ic_0 - ic_1\right)\frac{e^{\mu((i-1)\xi - (i-1)s_i)} - \left(i-1\right)\mu s_i e^{\mu((i-1)\xi - (i-1)s_i)}}{i} - \left(\lambda_1 - ic_0 - ic_1\right)\frac{\mu\xi e^{\mu(-\xi + s_i)}}{i} \\
&- \left(i-1\right)\mu\left(\lambda_1 - ic_0 - ic_1\right)\frac{e^{\mu((i-1)\xi - (i-1)s_i)}}{i^2\mu} - \left(\lambda_1 - ic_0 - ic_1\right)\frac{e^{\mu(-\xi + s_i)}}{i^2} \\
&+ c_1 e^{\mu((i-1)\xi - (i-1)s_i)} - \left(i-1\right)\mu\left(\lambda_0 + ic_1 s_i\right)\frac{e^{\mu((i-1)\xi - (i-1)s_i)}}{i} - c_1 e^{\mu(-\xi + s_i)} - \mu\left(\lambda_0 + ic_1 s_i\right)\frac{e^{\mu(-\xi + s_i)}}{i} \\
&+ c_0 e^{-(i-1)\mu(s_i - \xi)} - c_0\left(i-1\right)\mu s_i e^{-(i-1)\mu(s_i - \xi)} - c_0 e^{-(i-1)\mu(s_i - \xi)} + c_1 e^{\mu(-\xi + s_i)} + \mu\left(\lambda_0 + nc_1 s_i\right)\frac{e^{\mu(-\xi + s_i)}}{n} \\
&+ \mu\left(\lambda_1 - nc_0 - nc_1\right)\frac{\xi e^{\mu(-\xi + s_i)}}{n} + \left(\lambda_1 - nc_0 - nc_1\right)\frac{e^{\mu(-\xi + s_i)}}{n^2}.
\end{aligned}
$$

4 The main theorem

For our main results, we assume that the transaction fee rate λ_1 and the operational expenses rate c_1 are sufficiently large, which can formally be stated as follows:

(A-1) $\lambda_1 - c_0 > 0$, which implies

(A-1-1) $\lambda_1 > 0$.

(A-2) $-\dfrac{1}{n(n-1)}\lambda_1 + \dfrac{\lambda_0}{n\,\mathrm{BTI}} + \dfrac{c_0}{n-1} < c_1$.

(A-2) implies

(A-2-1) $\dfrac{1}{n} + \dfrac{(n-1)c_1 - c_0}{\lambda_1} > 0$.

Substituting ξ into s_i, we obtain

$$\begin{aligned}
\frac{\partial}{\partial s_i}U_i^s(\xi) &= \frac{1}{i^2n^2}\left[-i^2n(n-1)\mu\lambda_1\xi + i^2\lambda_1 + i^2n(n-1)c_1 - i^2nc_0 - i^2n(n-1)\mu\lambda_0\right] \\
&= \frac{i^2n(n-1)\mu\lambda_1}{i^2n^2}\left[-\xi + \frac{i^2\lambda_1}{i^2n(n-1)\mu\lambda_1} + \frac{i^2n(n-1)c_1}{i^2n(n-1)\mu\lambda_1} - \frac{i^2nc_0}{i^2n(n-1)\mu\lambda_1} - \frac{i^2n(n-1)\mu\lambda_0}{i^2n(n-1)\mu\lambda_1}\right] \\
&= \frac{(n-1)\mu\lambda_1}{n}\left[-\xi + \frac{1}{n(n-1)\mu} + \frac{c_1}{\mu\lambda_1} - \frac{c_0}{(n-1)\mu\lambda_1} - \frac{\lambda_0}{\lambda_1}\right] \\
&= \frac{(n-1)\mu\lambda_1}{n}\left[-\xi + \frac{\lambda_1 + n(n-1)c_1 - nc_0}{n(n-1)\mu\lambda_1} - \frac{\lambda_0}{\lambda_1}\right],
\end{aligned}$$

where $\lambda_1 > 0$ by **(A-1-1)**.

Proposition 8 Under **(A-1)**, we have $\frac{\partial^2}{\partial s_i^2}U_i^s(\xi) < 0$ for all i.

Lemma 9

$$\begin{aligned}
\frac{\partial^2}{\partial s_i^2}U_i^s(s_i,\xi_{-i})\,|_{s_i=\xi} = \frac{1}{n^2}\mu\big(&\lambda_1 - nc_0 - nc_1 - n^2\lambda_1 + n^2c_0 + (2-i)n^2c_1 + n\mu\lambda_0 \\
&- (2-i)n^2\mu\lambda_0 - (2-i)n^2\mu\xi\lambda_1 + n\mu\xi\lambda_1\big)
\end{aligned}$$

(This is correct for $n = 2$ and $i = 1$ as shown below)

Proof. $\left[\frac{1}{n^2}\mu\left(\lambda_1 - nc_0 - nc_1 - n^2\lambda_1 + n^2c_0 + (2-i)n^2c_1 + n\mu\lambda_0 - (2-i)n^2\mu\lambda_0 - (2-i)n^2\mu\xi\lambda_1 + n\mu\xi\lambda_1\right)\right]_{n=2}$

$= -\frac{1}{4}\mu\left(3\lambda_1 - 2c_0 - (6-4i)c_1 + (6-4i)\mu\lambda_0 + (6-4i)\mu\xi\lambda_1\right)$

$\left[-\frac{1}{4}\mu\left(3\lambda_1 - 2c_0 - (6-4i)c_1 + (6-4i)\mu\lambda_0 + (6-4i)\mu\xi\lambda_1\right)\right]_{i=1} = -\frac{1}{4}\mu\left(3\lambda_1 - 2c_0 - 2c_1 + 2\mu\lambda_0 + 2\mu\xi\lambda_1\right)$

$$\left[-\frac{1}{4}\mu\left(3\lambda_1-2c_0-2c_1+2\mu\lambda_0+2\mu\xi\lambda_1\right)\right]_{\xi=\frac{\lambda_1+2(c_1-c_0)-2\mu\lambda_0}{2\mu\lambda_1}}=-\frac{1}{4}\mu\left(4\lambda_1-4c_0\right)=\mu c_0-\mu\lambda_1=-\mu\left(\lambda_1-c_0\right)$$

■

Lemma 10

$$\left[\frac{1}{n^2}\mu\left(\lambda_1-nc_0-nc_1-n^2\lambda_1+n^2c_0+(2-i)\,n^2c_1+n\mu\lambda_0-(2-i)\,n^2\mu\lambda_0\right.\right.$$
$$\left.\left.-(2-i)\,n^2\mu\xi\lambda_1+n\mu\xi\lambda_1\right)\right]_{\xi=\frac{\lambda_1+n((n-1)c_1-c_0)-n(n-1)\mu\lambda_0}{n(n-1)\mu\lambda_1}}$$
$$=\frac{1}{n^2}\mu\left(\lambda_1-nc_0-nc_1-n^2\lambda_1-\frac{1}{n-1}\left(n\left(c_0-c_1\left(n-1\right)\right)-\lambda_1+n\mu\lambda_0\left(n-1\right)\right)+n^2c_0+(2-i)\,n^2c_1+n\mu\lambda_0\right.$$
$$\left.+(2-i)\,\frac{n}{n-1}\left(n\left(c_0-c_1\left(n-1\right)\right)-\lambda_1+n\mu\lambda_0\left(n-1\right)\right)-(2-i)\,n^2\mu\lambda_0\right)$$

Proof. $\frac{1}{n^2}\mu\left[\left(\lambda_1-nc_0-nc_1-n^2\lambda_1-\frac{1}{n-1}\left(n\left(c_0-c_1\left(n-1\right)\right)-\lambda_1+n\mu\lambda_0\left(n-1\right)\right)+n^2c_0+(2-i)\,n^2c_1+n\mu\lambda_0\right.\right.$
$\left.\left.+(2-i)\,\frac{n}{n-1}\left(n\left(c_0-c_1\left(n-1\right)\right)-\lambda_1+n\mu\lambda_0\left(n-1\right)\right)-(2-i)\,n^2\mu\lambda_0\right)\right]_{n=2}=-\frac{1}{4}\mu\left(\left(6-2i\right)\lambda_1-\left(8-4i\right)c_0\right)$

■

Lemma 11

$$\frac{1}{n^2}\mu\left(\lambda_1-nc_0-nc_1-n^2\lambda_1-\frac{1}{n-1}\left(n\left(c_0-c_1\left(n-1\right)\right)-\lambda_1+n\mu\lambda_0\left(n-1\right)\right)+n^2c_0+(2-i)\,n^2c_1+n\mu\lambda_0\right.$$
$$\left.+(2-i)\,\frac{n}{n-1}\left(n\left(c_0-c_1\left(n-1\right)\right)-\lambda_1+n\mu\lambda_0\left(n-1\right)\right)-(2-i)\,n^2\mu\lambda_0\right)<0$$

Proof. $\lambda_1-nc_0-nc_1-n^2\lambda_1-\frac{1}{n-1}\left(n\left(c_0-c_1\left(n-1\right)\right)-\lambda_1+n\mu\lambda_0\left(n-1\right)\right)+n^2c_0+(2-i)\,n^2c_1+n\mu\lambda_0$
$+(2-i)\,\frac{n}{n-1}\left(n\left(c_0-c_1\left(n-1\right)\right)-\lambda_1+n\mu\lambda_0\left(n-1\right)\right)-(2-i)\,n^2\mu\lambda_0$
$=-\frac{n}{n-1}\left(\left(1-i\right)\lambda_1-n\lambda_1+inc_0+n^2\lambda_1-n^2c_0\right)=-\frac{n}{n-1}\left(\left(n^2-n+\left(1-i\right)\right)\lambda_1-n\left(n-i\right)c_0\right)$. Thus
$\frac{1}{n^2}\mu\left(\lambda_1-nc_0-nc_1-n^2\lambda_1-\frac{1}{n-1}\left(n\left(c_0-c_1\left(n-1\right)\right)-\lambda_1+n\mu\lambda_0\left(n-1\right)\right)+n^2c_0+(2-i)\,n^2c_1+n\mu\lambda_0\right.$
$\left.+(2-i)\,\frac{n}{n-1}\left(n\left(c_0-c_1\left(n-1\right)\right)-\lambda_1+n\mu\lambda_0\left(n-1\right)\right)-(2-i)\,n^2\mu\lambda_0\right)$
$=-\frac{\mu}{n(n-1)}\left(\left(n^2-n+\left(1-i\right)\right)\lambda_1-n\left(n-i\right)c_0\right)$. Since $ni-n+(1-i)=(n-1)(i-1)\geq 0$, we obtain

$$\left(n^2-n+\left(1-i\right)\right)\lambda_1-n\left(n-i\right)c_0\geq n\left(n-i\right)\left(\lambda_1-c_0\right)>0$$

since $\lambda_1 - c_0 > 0$ by **(A-1)** and $n > i$. For $n = i$, we compute

$$\begin{aligned}
\left(n^2 - n + (1-i)\right)\lambda_1 - n(n-i)c_0 &= \left(n^2 - n + (1-n)\right)\lambda_1 \\
&= (n-1)^2 \lambda_1 > 0.
\end{aligned}$$

■

Thus

$$s_i^*(\mu) = \xi(\mu) = \frac{\lambda_1 + n(n-1)c_1 - nc_0}{n(n-1)\mu\lambda_1} - \frac{\lambda_0}{\lambda_1}$$

for all i is a Nash equilibrium given $\mu > 0$ for expected payoff functions $\{U_i^s\}$.

Theorem 12 We next compute

$$E[\mu, s; B] = \int_0^\infty t p_B(s, \mu; t)\, dt = \int_0^\infty e^{-\mu \sum_{j=1}^{n} (t-s_j)^+} dt.$$

Since $p_B(s, \mu; t)$ is symmetric in s, we may assume $s_1 \leq s_2 \leq \cdots \leq s_n$ to evaluate the above integral. We compute

$$\begin{aligned}
E[\mu, s; B] &= \int_0^\infty e^{-\mu\left[(t-s_1)^+ + \cdots + (t-s_n)^+\right]} dt \\
&= \int_0^{s_1} dt + \int_{s_1}^{s_2} e^{-\mu(t-s_1)} dt + \cdots + \int_{s_{n-1}}^{s_n} e^{-\mu[(t-s_1)+\cdots+(t-s_{n-1})]} dt + \int_{s_n}^\infty e^{-\mu[(t-s_1)+\cdots+(t-s_n)]} dt \\
&= s_1 + e^{\mu s_1}\frac{1}{\mu}\left(e^{-\mu s_1} - e^{-\mu s_2}\right) + e^{\mu(s_1+s_2)}\frac{1}{2\mu}\left(e^{-2\mu s_2} - e^{-2\mu s_3}\right) + \cdots + e^{\mu(s_1+\cdots+s_n)}\frac{1}{n\mu}e^{-n\mu s_n}
\end{aligned}$$

and obtain

$$E[\mu, (\xi(\mu), \cdots \xi(\mu)); B] = \xi(\mu) + \frac{1}{n\mu}.$$

Let $\mathrm{BTI} > 0$ denote the expected block time interval prescribed by the system. Since $\mu > 0$ is constrained by $\mathrm{BTI} = E[\mu, s; B]$, we must solve

$$\mathrm{BTI} = \frac{\lambda_1 + n(n-1)c_1 - nc_0}{n(n-1)\mu\lambda_1} - \frac{\lambda_0}{\lambda_1} + \frac{1}{n\mu}$$

for μ to obtain an equilibrium block finding rate μ^*.

Theorem 13 Under **(A-1)** and **(A-2)**, we obtain

$$\mu^* = \frac{1 + \frac{(n-1)c_1 - c_0}{\lambda_1}}{(n-1)\left(\mathrm{BTI} + \frac{\lambda_0}{\lambda_1}\right)} > 0$$

and

$$s_i^* = \xi(\mu^*) = \frac{\left(\mathrm{BTI} + \frac{\lambda_0}{\lambda_1}\right)\left\{1 + \frac{n[(n-1)c_1 - c_0]}{\lambda_1}\right\}}{n\left[1 + \frac{(n-1)c_1 - c_0}{\lambda_1}\right]} - \frac{\lambda_0}{\lambda_1} > 0.$$

Thus, (s^*, μ^*), where $s_i^* = \xi(\mu^*)$ for all i, constitutes a symmetric gap equilibrium for $\{U_i\}$, the original gap game.

Proof. Since **(A-2-1)** implies $0 < \frac{1}{n} + \frac{(n-1)c_1 - c_0}{\lambda_1} < 1 + \frac{(n-1)c_1 - c_0}{\lambda_1}$, we infer $\mu^* > 0$. By **(A-2)**, we have

$$\begin{aligned}
0 &< \frac{1}{n(n-1)}\lambda_1 + c_1 + \frac{c_0}{n-1} - \frac{\lambda_0}{n\,\mathrm{BTI}} \\
&= \frac{1}{n(n-1)}\frac{\lambda_1}{\mathrm{BTI}}\left[\mathrm{BTI}\left\{1 + \frac{n[(n-1)c_1 - c_0]}{\lambda_1}\right\} - \frac{\lambda_0}{\lambda_1}(n-1)\right] \\
&= \frac{1}{n(n-1)}\frac{\lambda_1}{\mathrm{BTI}}\left[\mathrm{BTI} + \frac{\lambda_0}{\lambda_1} + \mathrm{BTI}\,\frac{n[(n-1)c_1 - c_0]}{\lambda_1} - \frac{\lambda_0}{\lambda_1}n\right] \\
&= \frac{1}{n(n-1)}\frac{\lambda_1}{\mathrm{BTI}}\left[\left(\mathrm{BTI} + \frac{\lambda_0}{\lambda_1}\right) + \left(\mathrm{BTI} + \frac{\lambda_0}{\lambda_1}\right)\frac{n[(n-1)c_1 - c_0]}{\lambda_1} - \frac{\lambda_0}{\lambda_1}n - \frac{\lambda_0}{\lambda_1}\frac{n[(n-1)c_1 - c_0]}{\lambda_1}\right] \\
&= \frac{1}{n(n-1)}\frac{\lambda_1}{\mathrm{BTI}}\left[\left(\mathrm{BTI} + \frac{\lambda_0}{\lambda_1}\right)\left\{1 + \frac{n[(n-1)c_1 - c_0]}{\lambda_1}\right\} - \frac{\lambda_0}{\lambda_1}n\left\{1 + \frac{[(n-1)c_1 - c_0]}{\lambda_1}\right\}\right],
\end{aligned}$$

which proves $\xi(\mu^*) > 0$ as desired. ■

Corollary 14 Under **(A-1)** and **(A-2)**, we have

$$\lim_{n\to\infty} s_i^* = \mathrm{BTI}$$

for all i.

5 Conclusion

Although Tsabary and Eyal (2018) developed a complete symbolic description of gap games, they studied the equilibrium properties of these games primarily using numerical and simulation methods due to the complexity of expected payoff functions. We introduced a simplifying assumption where each miner controls a single mining rig with equal computational power, in contrast to Tsabary and Eyal's (2018) assumption that miners control multiple identical rigs.

This assumption greatly simplified the expression for individual temporal active rig shares, enabling us to compute the probability distribution function $p_B(s, \mu, t)$ of the global block finding time B clearly and rigorously in Section 3. We employed the typical assumption in the literature, where miners mine blocks as Bernoulli trials, and the time required for a successful result by each miner is a mutually independent random variable.

We may view a gap game as a family of games parametrized by the rate μ. The solution concept of Nash equilibrium applies to each game within this family, given a specific μ. We identified an appropriate notion of equilibrium for gap games; it is a Nash equilibrium denoted as $s^*(\mu)$, subject to the condition $\mathrm{BTI} = E[\mu, s; B]$.

In Section 4, we considered gap games in which both miners bear the same operational expenses, making them homogeneous. We observed that the symmetry of the miners leads to symmetric relations among their expected payoff functions. In our main theorem, we present explicitly a closed-form Nash-like solutions of the gap game under the assumption that the transaction fee rate and the operational expenses rate are sufficiently large.

While numerical methods and simulations are common in the current literature, our approach uses a proof-based game theoretic analysis to establish conditions for equilibrium existence across different reward and cost structures. We acknowledge that we limited the number of players to two to make our analysis mathematically tractable, but we intend to address this limitation in future research.

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6 Appendix 1-1

This section of the appendicitis explain how to describe a trial followed by the trials depending on the outcomes of the first trial. The method presented here can be found in Ito (1986) as a method referred to as tree composition trials. Tree connected trials are closely related to Young measures (transition probabilities) as briefly described below.

Let T_0 be a trial with probability space (Ω_0, P_0), where $\Omega_0 = \{a_1, a_2, \cdots, a_n\}$. For each $a \in \Omega_0$, let T_a be the trial with a probability space (Ω_a, P_a), where we may assume $\Omega_a = \Delta$ for all $a \in \Omega_0$ by taking $\Delta = \bigcup_{a \in \Omega_0} \Omega_a$. We consider the so-called tree composition trial T of T_0 followed by $\{T_a\}$. Consider the product trial $\widetilde{T} = T_0 \times T_{a_1} \times T_{a_2} \times \cdots \times T_{a_n}$ with the probability space $\left(\widetilde{\Omega}, \widetilde{P}\right)$, where $\widetilde{\Omega} = \Omega_0 \times \prod_{a \in \Omega_0} \Omega_a = \Omega_0 \times \Omega^n$ and $\widetilde{P} = P_0 \times \prod_{a \in \Omega_0} P_a$ (the product). Then, T is the mixture $\widetilde{T}_X$ of $\widetilde{T}$ by $X : \widetilde{\Omega} \to \Omega_0 \times \bigcup_{\omega \in \Omega_0} \Omega_\omega = \Omega_0 \times \Delta \equiv \Omega$, where $X\left(\widetilde{\omega}\right) = \left(\pi_0\left(\widetilde{\omega}\right), \pi_{\pi_0(\widetilde{\omega})}\left(\widetilde{\omega}\right)\right)$ with $\pi_0 : \widetilde{T} \to \Omega_0$ and $\pi_a : \widetilde{T} \to \Omega_a$ the projection maps, and $P = \widetilde{P}^X \equiv \widetilde{P} \circ X^{-1}$. Let $(a, b) \in \Omega = \Omega_0 \times \Delta$. We show that P can be described by a Young measure. We have a Young measure $P_\cdot : \Omega_0 \to \Pr\text{ob}\,(\Delta)$, defined by $a \mapsto P_a$. Let $A \times B \subset \Omega_0 \times \Delta$. Then

$$P_0 \otimes P_\cdot\,(A \times B) = \sum_{a \in A} P_a\,(B)\, P_0\,(A)\,.$$

Thus, if $(a, b) \in \Omega = \Omega_0 \times \Delta$, we obtain

$$\begin{aligned} P_0 \otimes P_\cdot\,(a, b) &= \sum_{a' \in \{a\}} P_{a'}\,(b)\, P_0\,(a') \\ &= P_0\,(a)\, P_a\,(b) \\ &= P\,(a, b)\,, \end{aligned}$$

as desired.

7 Appendix 1-2

In this section of the appendices, we address the problem of how to model the system process to choose a single successful miner from possibly multiple successful miners at a given time. We are concerned with the conditional probabilities of the successful mining of each miner, conditioned on the event that some miners successfully mine. In Tsabary, I., & Eyal, I. (2018, October), it is given as the ratio of the active rigs owned by miner i at time t to the total number of active rigs at time t. Thus, if each miner owns a single rig, as we assume for simplicity, then the ratio will be $\frac{1}{n}$, provided n is the number of miners and their rigs are all activated. The above ratio is simply taken as given in Tsabary, I., & Eyal, I. (2018, October) without probabilistic justification. We believe that deriving the ratio is highly non-trivial and it deserves fully rigorous consideration. In the sequel, we fix time t and assume first that all miners have stated their rigs before t, that is, $1 \leq s_i \leq t$ for all i. The basic idea is as follows: we view the mining at t as the first trial and if there are multiple successful miners, second trial follows to choose at random a single successful miner. We use the tree composition model to describe the outcomes of the first trial followed by the second.

Let $\Omega_0 = \{0,1\}$ and define a probability measure P_0 by $P_0(1) = p$ $(0 < p < 1)$. We consider the product space $(\Omega_0^\infty, P_0^\infty)$. We introduce the following notations to represents subsets of Ω_0^∞.

$$
\begin{aligned}
1 &\equiv \{1\} \times \Omega_0 \times \Omega_0 \times \cdots \subset \Omega_0^\infty \\
1^+ &\equiv \{0\} \times \Omega_0 \times \Omega_0 \times \cdots \subset \Omega_0^\infty \\
2 &\equiv \{(0,1)\} \times \Omega_0 \times \Omega_0 \times \cdots \subset \Omega_0^\infty \\
2^+ &\equiv \{(0,0)\} \times \Omega_0 \times \Omega_0 \times \cdots \subset \Omega_0^\infty \\
3 &\equiv \{(0,0,1)\} \times \Omega_0 \times \Omega_0 \times \cdots \subset \Omega_0^\infty \\
3^+ &\equiv \{(0,0,0)\} \times \Omega_0 \times \Omega_0 \times \cdots \subset \Omega_0^\infty \\
&\ddots
\end{aligned}
$$

Lemma 16 We have $P_0^\infty(0,0,0,\cdots) = 0$.

Proof. We have a decreasing sequence $1^+ \supset 2^+ \supset 3^+ \supset \cdots$, where $1^+ \cap 2^+ \cap 3^+ \cap \cdots = \{(0,0,0,\cdots)\} \subset \Omega_0^\infty$. Since $P_0^\infty(l^+) = q^l \to 0$ and P_0^∞ countably additive, $P_0^\infty(0,0,0,\cdots)$. ■

For each $y = (y_1, y_2, y_3, \cdots) \in \Omega_0^\infty$, if $y_1 = y_2 = \cdots = y_{l-1} = 0$ and $y_l = 1$ then let $1_y = l$, i.e., l is the location of the first 1 appeared after consecutive zeros in the components of y. Define $X_i : \Omega_0^\infty \diagdown \{(0,0,0,\cdots)\} \to \mathbb{N}$ by

$$X_i(y) = 1_y.$$

X_i is defined a.e. on Ω_0^∞ in view of the above lemma.

For each $i \in \{1, 2, \cdots, n\}$, define a conditional probability $P_{0,i}^\infty \in \Pr\text{ob}(\Omega_0^\infty)$ by

$$P_{0,i}^\infty(y) \equiv P_0^\infty(y \mid X_i \geq s_i).$$

Recalling that $\{X_i = t\} = t \subset \Omega_0^\infty$ and $\{X_i \geq s_i\} \subset \bigcup_{a \geq s_i} a \subset \Omega_0^\infty$, where $P_0^\infty\left(\bigcup_{a \geq s_i} a - \{X_i \geq s_i\}\right) = 0$ by the previous lemma, and $a \cap b = \varnothing$ if $a \neq b$, we obtain

$$\begin{aligned}
P_{0,i}^{\infty}(X_i = t) &= P_0^{\infty}(X_i = t \mid X_i \geq s_i) \\
&= P_0^{\infty}\left(t \mid \bigcup_{a \geq s_i} a\right) \\
&= P_0^{\infty}\left(t \cap \bigcup_{a \geq s_i} a\right) \diagup \sum_{a \geq s_i} P_0^{\infty}(a) \\
&= \begin{cases} 0 & t < s_i \\ q^{t-s_i} p & t \geq s_i \end{cases} .\text{(memorylessness)} \\
&= 1_{[s_i,\infty)}(t)\, q^{t-s_i} p, \text{ since } \sum_{a \geq s_i} P_0^{\infty}(a) = 1.
\end{aligned}$$

Define $\widetilde{P} \equiv \prod_{i=1}^{n} P_{0,i}^{\infty}$.

Lemma 17 We have $P_{0,i}^{\infty}(0, 0, 0, \cdots) = 0$ and hence $\widetilde{P}\left[(0, 0, 0, \cdots)^n\right] = 0$.

Proof. We have

$$\begin{aligned}
P_{0,i}^{\infty}(0, 0, 0, \cdots) &= P_0^{\infty}\left(\{(0, 0, 0, \cdots)\} \cap \bigcup_{a \geq s_i} a\right) \diagup \sum_{a \geq s_i} P_0^{\infty}(a) \\
&= P_0^{\infty}\left(\{(0, 0, 0, \cdots)\} \cap \bigcup_{a \geq s_i} a\right) \\
&= 0.
\end{aligned}$$

■

Regarding X_i as the mapping $X_i : \widetilde{\Omega} \to \mathbb{N}$, define

$$B = \min_{1 \leq i \leq n} \{X_i\} : \widetilde{\Omega} \to \mathbb{N},$$

which is defined a.e. and hence $\widetilde{P}^B \equiv \widetilde{P} \circ B^{-1} \in \Pr\text{ob}(\mathbb{N})$. Then,

$$B(x) = t \text{ iff } \min_{1 \leq k \leq n} \{1_{x_k}\} = t.$$

Let $x = (x_1, x_2, \cdots, x_n) \in \widetilde{\Omega}$. Define a Young measure $P : \widetilde{\Omega} \to \{1, 2, \cdots, n\}$ by

$$P_x(i) = \frac{1}{|\min_{1\leq k\leq n}\{1_{x_k}\}|} \text{ if } 1_{x_i} \in \min_{1\leq k\leq n}\{1_{x_k}\}.$$

Then,

$$P_x(i) = 0 \text{ if } 1_{x_i} \notin \min_{1\leq k\leq n}\{1_{x_k}\}.$$

Let $a_1 \times a_2 \times \cdots \times a_n \subset \widetilde{\Omega}$. Define

$$t_{i_1 i_2 \cdots i_k} \equiv a_1 \times a_2 \times \cdots \times a_n,$$

where $a_{i_1} = a_{i_2} = \cdots = a_{i_k} = t$, and $a_i = t^+$ otherwise.

Lemma 18 If $t_{i_1 i_2 \cdots i_k} \neq t_{j_1 j_2 \cdots j_l}$ then $t_{i_1 i_2 \cdots i_k} \cap t_{j_1 j_2 \cdots j_l} = \varnothing$.

Proof. Let $t_{i_1 i_2 \cdots i_k} = a_1 \times a_2 \times \cdots \times a_n$ and $t_{j_1 j_2 \cdots j_l} = b_1 \times b_2 \times \cdots \times b_n$. Then, there must be i such that $a_i = t$ and $b_i = t^+$, or the other way around. Then,

$$\begin{aligned}\left(\prod_{k=1}^{n} a_k\right) \cap \left(\prod_{k=1}^{n} b_k\right) &= \prod_{k=1}^{n} a_k \cap b_k \\ &= \varnothing\end{aligned}$$

since $a_i \cap b_i = t \cap t^+ = \varnothing$. ■

Regarding B as a mapping $B : \widetilde{\Omega} \times \{1, 2, \cdots, n\} \to \mathbb{N}$, define $B_i : \widetilde{\Omega} \to \mathbb{N}$ by $B_i(x) \equiv B(x, i)$. Then

$$\begin{aligned}\{B = t\} &= \bigcup_{1\leq i_1 \cdots i_k \leq n} t_{i_1 i_2 \cdots i_k} \times \{1, 2, \cdots, n\}, \text{ and} \\ \{B_i = t\} &= \bigcup_{1\leq i_1 \cdots i_k \leq n} t_{i_1 i_2 \cdots i_k} \times \{i\}.\end{aligned}$$

Lemma 19 $\sum_{x \in t_{i_1 i_2 \cdots i_k}} \widetilde{P}(x) P_x(i) = \widetilde{P}(t_{i_1 i_2 \cdots i_k}) P_{t_{i_1 i_2 \cdots i_k}}(i)$, where $P_{t_{i_1 i_2 \cdots i_k}}$ is well-defined.

Proof. Let $x \in t_{i_1 i_2 \cdots i_k}$. Then, $x_{i_1} \in t, \cdots, x_{i_k} \in t$ and $x_i \in t^+$ for all $i \notin \{i_1, \cdots, i_k\}$. Hence, $1_{x_{i_1}} = t, \cdots, 1_{x_{i_k}} = t$ and $1_{x_i} \geq t + 1$ for all $i \notin \{i_1, \cdots, i_k\}$. Thus, $\min_{1\leq k\leq n}\{1_{x_k}\} = t$ and $|\min_{1\leq k\leq n}\{1_{x_k}\}| = k$ and we deduce

$$P_x(i) = \frac{1}{k} \epsilon^i_{t_{i_1 i_2 \cdots i_k}},$$

where

$$\epsilon^i_{t_{i_1 i_2 \cdots i_k}} = \begin{cases} 1 & i \in \{i_1, \cdots, i_k\} \\ 0 & \text{otherwise} \end{cases}.$$

Thus, $P_{t_{i_1 i_2 \cdots i_k}}(i)$ is well-defined. Then,

$$\begin{aligned} \sum_{x \in t_{i_1 i_2 \cdots i_k}} \widetilde{P}(x) P_x(i) &= \sum_{x \in t_{i_1 i_2 \cdots i_k}} \widetilde{P}(x) P_{t_{i_1 i_2 \cdots i_k}}(i) \\ &= \widetilde{P}(t_{i_1 i_2 \cdots i_k}) P_{t_{i_1 i_2 \cdots i_k}}(i). \end{aligned}$$

■

Let $s = s_1 + \cdots + s_n$. We compute

$$\begin{aligned} P(\{B_i = t\} \cap \{B = t\}) &= P(\{B_i = t\}) \\ &= \sum_{1 \le i_1 \cdots i_k \le n} \widetilde{P}(t_{i_1 i_2 \cdots i_k}) P_{t_{i_1 i_2 \cdots i_k}}(i) \\ &= \sum_{1 \le i_1 \cdots i_k \le n} \frac{1}{k} \epsilon^i_{t_{i_1 i_2 \cdots i_k}} \prod_{i \in \{i_1, \cdots, i_k\}} q^{t - s_i} p \prod_{i \notin \{i_1, \cdots, i_k\}} q^{t - s_i + 1} \\ &= \sum_{k=1}^{n} \frac{1}{k} \binom{n-1}{k-1} p^k q^{nt - k + (n-s)} \\ &= \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} p^k q^{nt - k + (n-s)} \\ &= \frac{1}{n} P(\{B = t\}). \end{aligned}$$

We conclude that

$$P(B_i = t \mid B = t) = \frac{1}{n}.$$

8 Appendix 1-3

In this last section of the appendices, we first prove the crucial fact that is $P(B = t) > 0$ then some miner i must have started the rig no later than t, i.e., $s_i \le t$, and we lastly demonstrate that how the general case ($1 \le s_i$ for all i) reduces to the former case ($1 \le s_i \le t$ for all i) to complete the proof for our main assertion.

Proposition 20 We have the identity

$$P(B=t)=\begin{cases} 1-q^{\sum_{i=1}^{n}(2-s_i)^+} & t=1 \\ q^{\sum_{i=1}^{n}(t-s_i)^+}-q^{\sum_{i=1}^{n}(t+1-s_i)^+} & t>1 \end{cases}.$$

Proof. Recalling that

$$P_{0,i}^{\infty}(t)=1_{[s_i,\infty)}(t)\,q^{t-s_i}p,$$

we compute

$$\begin{aligned} P(B_i>t) &= \textstyle\sum_{u=t+1}^{\infty}1_{[s_i,\infty)}(u)\,q^{u-s_i}p \\ &= \textstyle\sum_{u=\max\{t+1,s_i\}}^{\infty}q^{u-s_i}p \\ &= q^{\max\{t+1,s_i\}-s_i}p\left(1+q+q^2+\cdots\right) \\ &= q^{\max\{t+1,s_i\}-s_i} \\ &= q^{(t+1-s_i)^+} \end{aligned}$$

and hence

$$\begin{aligned} P(B\geq t+1) = P(B>t) & \\ &= \textstyle\prod_{i=1}^{n}P(B_i>t) \\ &= \textstyle\prod_{i=1}^{n}q^{(t+1-s_i)^+} \\ &= q^{\sum_{i=1}^{n}(t+1-s_i)^+}. \end{aligned}$$

Thus

$$\begin{aligned} P(B=t) &= \begin{cases} 1-P(B>1) & t=1 \\ [1-P(B>t)]-[1-P(B>t-1)] & t>1 \end{cases} \\ &= \begin{cases} 1-q^{\sum_{i=1}^{n}(2-s_i)^+} & t=1 \\ q^{\sum_{i=1}^{n}(t-s_i)^+}-q^{\sum_{i=1}^{n}(t+1-s_i)^+} & t>1 \end{cases}. \end{aligned}$$

■

Corollary 21 $P(B=t)>0$ if and only if some miner i starts the rig no later than t, i.e., $s_i\geq t$.

Proof. Suppose $t = 1$. Then, $P(B = 1) > 0$ iff $1 - q^{\sum_{i=1}^{n}(2-s_i)^+} > 0$ iff $\sum_{i=1}^{n}(2-s_i)^+ > 0$ iff $(2-s_i)^+ > 0$ for some i iff $2 - s_i > 0$ for some i iff $s_i = 1$ for some i. Next, suppose $t > 1$. Then, $P(B = t) > 0$ iff $q^{\sum_{i=1}^{n}(t-s_i)^+} - q^{\sum_{i=1}^{n}(t+1-s_i)^+} > 0$ iff $\sum_{i=1}^{n}\left[(t+1-s_i)^+ - (t-s_i)^+\right] > 0$ iff $(t+1-s_i)^+ > (t-s_i)^+$ for some i iff $t - s_i = 0, 1, 2, \cdots$ for some i iff $t \geq s_i$ for some i. ■

Define

$$\begin{aligned} N(t) &\equiv \{i : s_i \geq t\} \\ &= \left\{i : 1_{[s_i,\infty)}(t) = 1\right\} \\ n(t) &\equiv |N(t)| \\ &= \sum_{i=1}^{n} 1_{[s_i,\infty)}(t). \end{aligned}$$

The previous corollary implies that if $P(B = t) > 0$ then $N(t) \neq \varnothing$. From the previous section, we infer for each $i \in N(t)$,

$$P(B_i = t \mid B = t) = \frac{1}{n(t)}$$

and hence for all i, we obtain

$$P(B_i = t \mid B = t) = \frac{1_{[s_i,\infty)}(t)}{\sum_{i=1}^{n} 1_{[s_i,\infty)}(t)},$$

provided $P(B = t) > 0$.

Appendix 2

We prove by induction on n. We compute

$$\begin{aligned}
\sum_{i=1}^{n+1} 1_{[s_i,\infty)} &= \sum_{i=1}^{n-1} i \cdot 1_{[s_i,s_{i+1})} + n \cdot 1_{[s_n,\infty)} + 1_{[s_{n+1},\infty)} \\
&= \sum_{i=1}^{n-1} i \cdot 1_{[s_i,s_{i+1})} + n \left[1_{[s_n,\infty)} + 1_{[s_{n+1},\infty)}\right] - n \cdot 1_{[s_{n+1},\infty)} + 1_{[s_{n+1},\infty)} \\
&= \sum_{i=1}^{n-1} i \cdot 1_{[s_i,s_{i+1})} + n \left[1_{[s_n,s_{n+1})} + 2 \cdot 1_{[s_{n+1},\infty)}\right] - n \cdot 1_{[s_{n+1},\infty)} + 1_{[s_{n+1},\infty)} \\
&= \sum_{i=1}^{n} i \cdot 1_{[s_i,s_{i+1})} + 2n \cdot 1_{[s_{n+1},\infty)} - n \cdot 1_{[s_{n+1},\infty)} + 1_{[s_{n+1},\infty)} \\
&= \sum_{i=1}^{n} i \cdot 1_{[s_i,s_{i+1})} + (n+1) \cdot 1_{[s_{n+1},\infty)},
\end{aligned}$$

as desired.

$$
\begin{aligned}
E[\mu, s; B] &= \int_0^\infty t p_B(s,\mu;t)\,dt \\
&= \int_0^\infty \mu \sum_{i=1}^n 1_{[s_i,\infty)}(t)\, t \cdot e^{-\mu \sum\limits_{j=1}^n (t-s_j)^+} dt \\
&= \int_0^\infty \mu \left[\sum_{i=1}^{n-1} i \cdot 1_{[s_i,s_{i+1})}(t) + n \cdot 1_{[s_n,\infty)}(t) \right] t \cdot e^{-\mu \sum\limits_{j=1}^n (t-s_j)^+} dt \\
&= \mu \sum_{i=1}^{n-1} i \int_0^\infty 1_{[s_i,s_{i+1})}(t)\, t \cdot e^{-\mu \sum\limits_{j=1}^i (t-s_j)} dt + \mu n \int_0^\infty 1_{[s_n,\infty)}(t)\, t \cdot e^{-\mu \sum\limits_{j=1}^n (t-s_j)} dt \\
&= \mu \sum_{i=1}^{n-1} i \int_{s_i}^{s_{i+1}} t e^{-\mu \sum\limits_{j=1}^i (t-s_j)} dt + \mu n \int_{s_n}^\infty t e^{-\mu \sum\limits_{j=1}^n (t-s_j)} dt \\
&= \sum_{i=1}^{n-1} \left[s_i e^{-\mu \sum\limits_{j=1}^i (s_i-s_j)} - s_{i+1} e^{-\mu \sum\limits_{j=1}^i (s_{i+1}-s_j)} \right] + \sum_{i=1}^{n-1} \int_{s_i}^{s_{i+1}} e^{-\mu \sum\limits_{j=1}^i (t-s_j)} dt + \mu n \int_{s_n}^\infty t e^{-\mu \sum\limits_{j=1}^n (t-s_j)} dt \\
&= \sum_{i=1}^{n-1} \left[s_i e^{-\mu \sum\limits_{j=1}^i (s_i-s_j)} - s_{i+1} e^{-\mu \sum\limits_{j=1}^i (s_{i+1}-s_j)} \right] + \sum_{i=1}^{n-1} \int_{s_i}^{s_{i+1}} e^{-\mu \sum\limits_{j=1}^{n-1} (t-s_j)^+} dt + \mu n \int_{s_n}^\infty t e^{-\mu \sum\limits_{j=1}^n (t-s_j)} dt \\
&= \sum_{i=1}^{n-1} \left[s_i e^{-\mu \sum\limits_{j=1}^i (s_i-s_j)} - s_{i+1} e^{-\mu \sum\limits_{j=1}^i (s_{i+1}-s_j)} \right] + \int_{s_1}^{s_n} e^{-\mu \sum\limits_{j=1}^{n-1} (t-s_j)^+} dt + \mu n \int_{s_n}^\infty t e^{-\mu \sum\limits_{j=1}^n (t-s_j)} dt \\
&= \sum_{i=1}^{n-1} \left[s_i e^{-\mu \sum\limits_{j=1}^i (s_i-s_j)} - s_{i+1} e^{-\mu \sum\limits_{j=1}^i (s_{i+1}-s_j)} \right] + \int_{s_1}^{s_n} e^{-\mu \sum\limits_{j=1}^{n-1} (t-s_j)^+} dt + s_n e^{-\mu \sum\limits_{j=1}^n (s_n-s_j)} + \int_{s_n}^\infty e^{-\mu \sum\limits_{j=1}^n (t-s_j)} dt \\
&= \sum_{i=1}^{n-1} \left[s_i e^{-\mu \sum\limits_{j=1}^i (s_i-s_j)} - s_{i+1} e^{-\mu \sum\limits_{j=1}^i (s_{i+1}-s_j)} \right] + s_n e^{-\mu \sum\limits_{j=1}^n (s_n-s_j)} + \int_{s_1}^{s_n} e^{-\mu \sum\limits_{j=1}^{n-1} (t-s_j)^+} dt + \int_{s_n}^\infty e^{-\mu \sum\limits_{j=1}^n (t-s_j)^+} dt \\
&= \sum_{i=1}^{n-1} \left[s_i e^{-\mu \sum\limits_{j=1}^i (s_i-s_j)} - s_{i+1} e^{-\mu \sum\limits_{j=1}^i (s_{i+1}-s_j)} \right] + s_n e^{-\mu \sum\limits_{j=1}^n (s_n-s_j)} + \int_{s_1}^\infty e^{-\mu \sum\limits_{j=1}^n (t-s_j)^+} dt \\
&= s_1 + \int_{s_1}^\infty e^{-\mu \sum\limits_{j=1}^n (t-s_j)^+} dt \\
&= \int_0^\infty e^{-\mu \sum\limits_{j=1}^n (t-s_j)^+} dt,
\end{aligned}
$$

since

$$
\int_0^{s_1} e^{-\mu \sum\limits_{j=1}^n (t-s_j)^+} dt = s_1
$$

and

$$\sum_{i=1}^{n-1}\left[s_i e^{-\mu\sum\limits_{j=1}^{i}(s_i-s_j)} - s_{i+1}e^{-\mu\sum\limits_{j=1}^{i}(s_{i+1}-s_j)}\right] + s_n e^{-\mu\sum\limits_{j=1}^{n}(s_n-s_j)}$$

$$= s_1.$$